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MATHEMATICAL LAWS FOR THE STEADY STATE AND COOLING IN A METALLIC BAR

JUAN J. MORALES

Departamento de Física, Facultad de Ciencias, Universidad de Extremadura, 06071 Badajoz (Spain)

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ABSTRACT

A study of the best governing equation for the linear flow of heat in a metallic bar has been made. The mathematical solutions for the steady-state temperature distribution and for the cooling are not unique. Three different equations describing the steady state were found for all the cases studied. These equations agree within the statistical error. For the cooling, two types of solutions have been found which depend on the temperature gradient in the bar.

INTRODUCTION

In liquids and gases the transfer processes of conduction, convection and radiation can occur simultaneously. In transparent solids, conduction and radiation transfer can both occur, while the flow of heat in opaque solids takes place exclusively by conduction. The physical phenomena, the fundamental laws, the thermophysical properties and the characteristic mathematical formulations of the process of heat transfer in matter are topics which have been reported in the literature [1,2]. However, temperature distributions, heat rates, temperature-time histories, etc., are generally complicated and in many cases quite difficult to deal with. To overcome these difficulties, physical and geometrical simplifications are used to obtain useful results. Thus, although it is known in advance that the thermophysical properties depend on the temperature, it is a reasonable approximation to consider them as being independent of temperature whenever the temperature variation is not large, or whenever the physical problem involves no phase changes or chemical reactions. Other simplifications derive from the geometry of the problem: for example, when the diameter of a rod is so small as compared with its length that there will be no radial temperature distribution, but there will be a large axial temperature distribution, etc.

In some cases the mathematical model chosen to treat the problem does not work because the simplifications used are not optimal, but a different

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method can be found to explain the same problem that gives the correct results. Therefore, the mathematical equations can be different depending on the particular reasonable approximation taken.

In this study, starting from the temperature distribution in a metallic bar for (i) the steady state and (ii) its cooling, we have tried to fit these experimental data to some mathematical equations using the appropriate statistical methods [3,4], in order to obtain the optimal governing equations.

FORMULATION OF THE GOVERNING EQUATIONS

We consider a bar characterized by a constant area of cross-section w, length L, perimeter p, conductivity K, density ρ, specific heat c, diffusivity κ and surface conductance H. The general method is to consider the case where the cross-section of the bar is smaller than its length, so that the flow of heat is predominantly in one direction, there being small losses in the perpendicular direction.

If the bar lies along the x axis, the heat gain by flow between the two faces of an element in x and $x + \mathrm{d}x$ is

$$wK\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\mathrm{d}x \tag{1}$$

where θ is the temperature.

The heat lost by radiation at the lateral surface is given by the Newton cooling law

$$H(\theta - \theta_0)p\ \mathrm{d}x \tag{2}$$

where θ_0 is the temperature of the medium into which the bar radiates.

The total gain of heat in the element is

$$wc\rho\frac{\partial\theta}{\partial t}\mathrm{d}x \tag{3}$$

Balancing eqns. (1) to (3) and setting $\nu = Hp/c\rho w$ and $\kappa = K/\rho c$ gives

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial x^2} - \nu(\theta - \theta_0) \tag{4}$$

Equation (4) permits two cases to be differentiated: (i) when there is radiation into a medium at constant temperature, and (ii) when there is no radiation. In the first case eqn. (4) takes the form

$$\frac{\partial\theta}{\partial t} = \kappa\frac{\partial^2\theta}{\partial x^2} \tag{5}$$

and the problem of the distribution of temperature in the bar is reduced to one of linear flow.

In the second case, the constant temperature of the medium may be taken as the zero of our scale and, if we make the change of variable $\theta = u \exp(-\nu t)$, eqn. (4) reduces to

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{6}$$

which is similar to eqn. (5).

A number of simple solutions of eqn. (4) are reviewed here and are applied to the problem under study [5].

Steady state

The steady state is the particular case when the flow of heat is invariant with time. Then eqn. (4) becomes

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{7}$$

where $m^2 = Hp/Kw$.

The general solution [6] of eqn. (7) has the form

$$\theta = \theta_1 e^{-mx} + \theta_2 e^{+mx} \tag{8}$$

where θ_1 and θ_2 are constants that must be calculated from the suitable boundary conditions.

(a) If the bar is considered semi-infinite, the ends are at constant temperatures θ_1 ($x = 0$) and $\theta_2 = 0$ ($L = \infty$), and eqn. (8) reduces to

$$\theta = \theta_1 e^{-mx} \tag{9}$$

(b) If the bar is finite with θ_1 at $x = 0$ and θ_2 at $x = L$ the solution of eqn. (8) will be

$$\theta = \frac{\theta_1 \sinh[m(L-x)] + \theta_2 \sinh(mx)}{\sinh(mL)} \tag{10}$$

(c) If there is no flow of heat from the end $x = L$ of the bar ($\theta_2 = 0$) we have in place of eqn. (10)

$$\theta = \theta_1 \frac{\cosh[m(L-x)]}{\cosh(mL)} \tag{11}$$

Cooling

Once the steady state has been reached, it is attempted to fit the data obtained from the cooling of the bar to the best solution of eqn. (4). The common procedure is to consider the cooling as an infinite series of exponential decay with time, but in practice the problem is to know when the series converges, i.e. how many terms must be taken to fit our experimental results.

Exponential decay is not the only solution to the problem because other theories can be applied to the cooling, such as the theory which considers that the temperature decay with time is proportional to the inverse of the fourth power of the temperature.

Both these methods were used in this study and the results from them compared.

EXPERIMENTAL PROCEDURE AND RESULTS

The study object used was a cylindric iron bar with the physical and geometrical characteristics shown in Table 1. As the cylinder has a small cross-section compared with its length, the problem is one of linear flow in which the temperature is specified by the time and the distance x measured along the rod. To measure the temperature, several thin holes were drilled perpendicularly from the generatrix to the axis, where chromel–alumen thermopars were placed and connected to a digital thermometer. The distances between thermopars are shown in Table 2 (first column) having been chosen taking into account the temperature gradient. The first thermopar, taken as the coordinate origin, was 11 cm from the first end of the bar.

TABLE 1

The physical and geometrical characteristics of the iron bars studied

$\rho = (7.8 \pm 0.1) \times 10^3$ kg m^{-3}
$c = (4.52 \pm 0.01) \times 10^2$ J kg^{-1} K^{-1}
$K = (80.3 \pm 0.1)$ J s^{-1} K^{-1}
$\kappa = (2.28 \pm 0.01) \times 10^{-5}$ m^2 s^{-1}
$L = (155 \pm 0.1) \times 10^{-2}$ m
$d = (550.0 \pm 0.5) \times 10^{-4}$ m
$p = (172.8 \pm 0.1) \times 10^{-3}$ m
$w = (23.8 \pm 0.1) \times 10^{-4}$ m^2

TABLE 2

The initial conditions used to obtain the three steady states I–III

Steady state	Potential difference, V (V)	Intensity, I (A)	Time (h)
I	45	0.35	4
II	58	0.47	5
III	80	0.65	12

TABLE 3

The temperature distributions corresponding to the three steady states I–III

X ($\times 10^{-2}$ m)	θ_{I}	θ_{II}	θ_{III}
0.0	25.4	34.9	61.5
4.0	21.9	29.9	53.1
10.0	17.4	24.3	42.8
16.0	13.7	19.4	34.7
25.5	9.4	13.6	25.0
36.0	6.2	9.3	17.5
51.0	3.3	5.2	10.4
66.0	1.5	2.7	6.2
81.0	0.0	1.5	3.7
96.0	0.0	0.8	2.2
116.0	0.0	0.6	1.2
136.0	0.0	0.4	0.7

The bar was heated from the end by a resistive coil connected to an a.c. variable transformer. The bar was held on two wooden supports and isolated by means of asbestos cord. The room temperature was taken from a thermopar inserted from the generatrix to the centre of a cylindrical copper block (5 cm high, 9 cm diameter).

Steady state

Table 2 lists the initial conditions for the three (I–III) steady-state cases studied. The last column in Table 2 gives the time spent in reaching the steady state, and V and I are the conditions used to heat the bar. The final values of the temperature are shown in Table 3, where θ is the temperature excess with respect to the medium. Equations (9)–(11) were used to fit these data. The results of the fits for the first steady state θ_{I} are given in Table 4, where σ is the standard deviation and r^2 is the correlation coefficient. (The correlation coefficient, r^2, gives the goodness of fit of the experimental data to a curve. If $r^2 = 1$ the curve is perfect. The values of r^2 are between 0 and 1.) As can be seen from Table 4 the three different equations give the same

TABLE 4

Values of the fit corresponding to the first steady state I

Equation	$\theta_1 \pm \sigma$	$m \pm \sigma$	r^2
9	25.7 ± 0.6	4.0 ± 0.06	0.99
10	25.7 ± 0.6	4.0 ± 0.1	0.99
11	25.6 ± 0.7	4.0 ± 0.2	0.99

TABLE 5

The fit for the steady states II and III using eqns. (9)–(11)

Steady state	$\theta_1 \pm \sigma$	$m \pm \sigma$	r^2
II	34.9 ± 0.7	3.7 ± 0.2	0.99
III	61.5 ± 0.7	3.5 ± 0.2	0.99

results within the statistical error, but the tendency is that eqn. (9) is more accurate, corresponding to a single exponential decay with distance x. Equation (9) has the advantage that it is independent of the bar length L, in contrast with the other equations.

The same procedure was carried out for the other two steady states and the results were qualitatively the same. The final mean values for cases II and III from fitting to eqns. (9)–(11) are given in Table 5, where the standard deviation and the correlation coefficient are as in case I.

In general, when a physics problem may be explained by several methods, the easiest solution is chosen. This is the reason why, for this range of temperature, the exponential decay with distance x for the iron bar is usually chosen. To illustrate the exponential behaviour the last steady state III is plotted in Fig. 1. The curve represents the temperature distribution (left ordinate) versus distance. When the logarithmic temperature (right ordinate) is plotted versus distance, a straight line is obtained which verifies the exponential decay for the temperature.

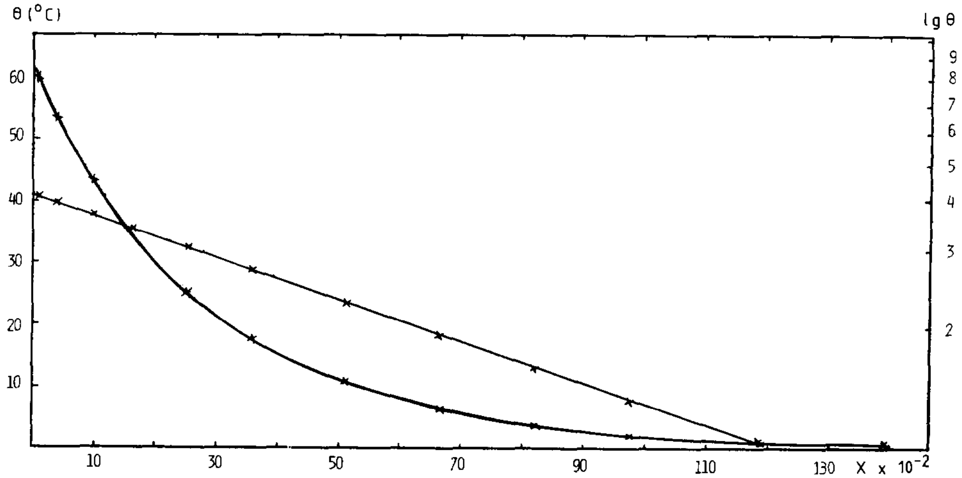

Fig. 1. Temperature distribution for steady state III. Verification of the exponential behaviour.

Cooling

Starting from the temperature distribution in the steady state, the cooling was performed by switching off the heater and removing it from the end of the bar in order to avoid the thermal inertia of the heater. The first temperatures were taken after 1 min and the other measurements were done every 13 min until 320 min when the temperatures in the bar were seen to be near room temperature.

The result presented here is the cooling starting from steady state III. The temperatures θ_1, θ_2, $\theta_3, \ldots, \theta_i$ are those at $x = 0.0$, 4.0, 10.0,... x_i (see Table 3), such that θ_i is the cooling temperature distribution at the point i distant x_i from the first point.

Table 6 lists the results for the first six points in the bar. The standard deviations for the constants are ± 1 and for the exponents ± 0.02. The correlation coefficients for these results are $r^2 = 0.99$ in all cases. When the temperature is higher the temperature is much better fitted by a double exponential, because the first exponential governs the faster decay of the temperature at the start of cooling, while the second exponential governs the decay for long times after the cooling has begun and when the temperature excess is not so large. If the starting cooling temperature is lower, the first exponential decreases compared with the second exponential, and, when the difference between the bar and the room temperatures is not too high, the cooling can be fitted with a single exponential (θ_6) until the points are reached where the bar is not affected by the temperature gradient and the temperature is already constant with time and close to the room temperature. Figure 2 shows the results obtained for the cooling of the bar, and it can be seen how the thermal inertia of the bar manifests itself when the temperature excess is small.

The cooling data were not only fitted to the single and double exponential equations, but also to the power decay law. Figure 3 shows the plot of the inverse fourth power of the temperature against time for the first point in the bar (θ_1). Figure 3 shows the linear behaviour of the cooling, the fit being the line $\theta^{-1/4} = 42t + 0.36$, with $r^2 = 0.99$. When other points in the bar

TABLE 6

The fit for the cooling starting from steady state III

θ_1	$36e^{-0.46t} + 25e^{-0.14t}$
θ_2	$34e^{-0.40t} + 25e^{-0.13t}$
θ_3	$27e^{-0.43t} + 25e^{-0.13t}$
θ_4	$18e^{-0.40t} + 22e^{-0.13t}$
θ_5	$15e^{-0.35t} + 19e^{-0.12t}$
θ_6	$19e^{-0.13t}$

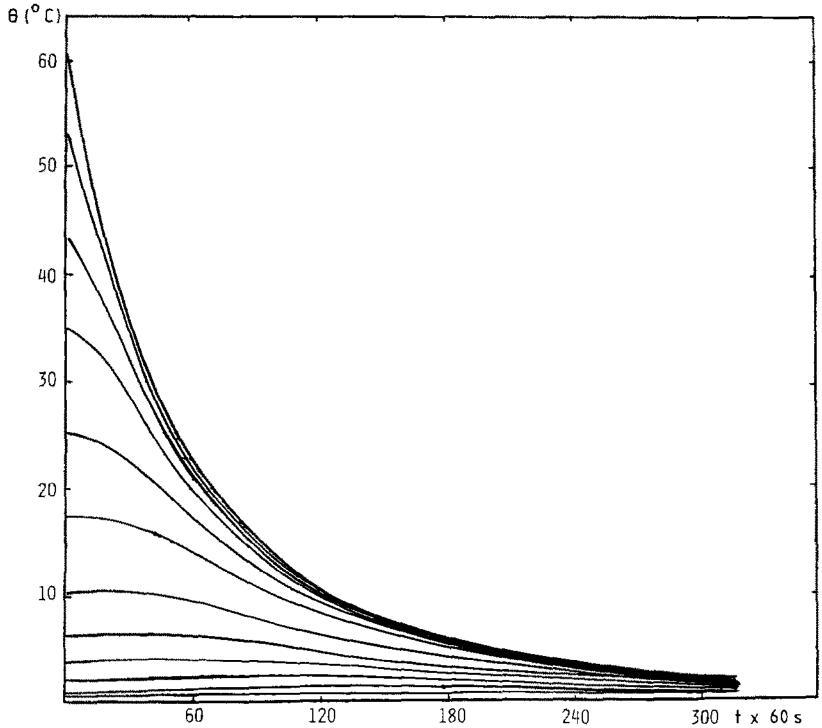

Fig. 2. Cooling curves for each point in the iron bar. The cooling was started from steady state III.

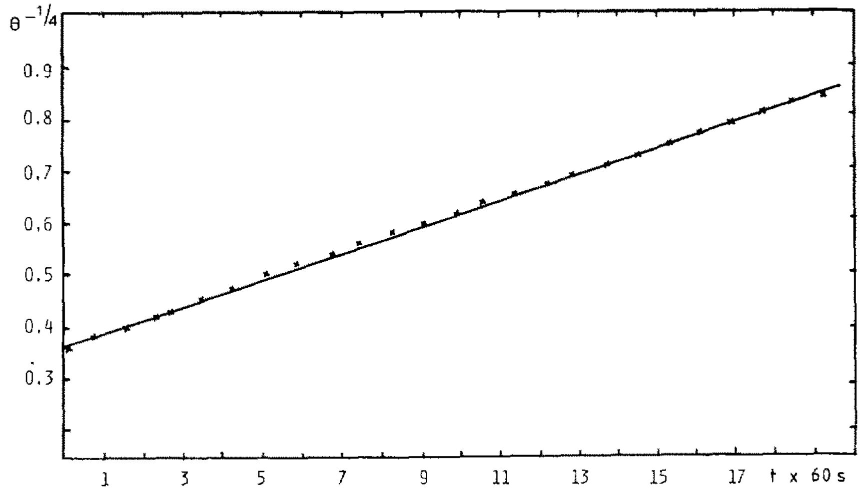

Fig. 3. Linear time behaviour of the cooling for the inverse of the fourth power of the temperature.

(θ_2, $\theta_3, \ldots$) were taken, the fits to the linear regression proportional to $\theta^{-1/4}$ worsened because the temperature excess was smaller.

Another kind of fit, as a single potential decay, was tried but was worse in all cases.

The same studies were carried out for the cooling of the other steady states I and II and the results were qualitatively the same.

CONCLUSIONS

In theory, the solution of the heat transfer in a material with some given boundary condition is unique. In practice, this is not exactly true because some factors affect the uniqueness of the solutions. These factors are related to the experimental procedure, the quality and geometry of the materials and, most of all, to the temperature range [7]. From the results obtained it is concluded that the governing equation for temperature distribution in the steady state and in the cooling process is not unique.

For the example described in this paper, there are three equations for the steady state which each have a different meaning: a semi-infinite bar, eqn. (9); a finite bar with flow of heat from the end, eqn. (10); and a finite bar without flow of heat from the end, eqn. (11). These equations give the same results within the statistical error, and permit the temperature distribution in the bar to be explained.

The cooling experiment showed how the temperature affects the governing equations. When the temperature difference between any point of the bar and the room are relatively high, the best fit is a double exponential; when the differences are not so high, the equation is a single exponential; and when the differences are relatively small, the exponential becomes a linear behaviour with time. However, the exponential decay is not the unique solution of the problem either, because the inverse fourth power law turns out to be valid when the temperature differences between the bar and the room are relatively high. Thus, the solutions of the cooling depend on the temperatures of the steady state from which the cooling is started.

These are the results for an iron bar, which we considered, in principle, to be the easiest object to study. When the same mathematical analysis is made for a bad conductor and a transparent medium, such as poly(methyl methacrylate) (PMMA), the results are quite different, as will be shown in a forthcoming paper.

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